

# An intrinsic characterization of 2+2 warped spacetimes

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**Abstract.** We give several equivalent conditions that characterize the 2+2 warped spacetimes: imposing the existence of a Killing-Yano tensor  $A$  subject to complementary algebraic restrictions; in terms of the projector  $v$  (or of the canonical 2-form  $U$ ) associated with the 2-planes of the warped product. These planes are principal planes of the Weyl and/or Ricci tensors and can be explicitly obtained from them. Therefore, we obtain the necessary and sufficient (local) conditions for a metric tensor to be a 2+2 warped product. These conditions exclusively involve explicit concomitants of the Riemann tensor. We present a similar analysis for the conformally 2+2 product spacetimes and give an invariant classification of them. The warped products correspond to two of these invariant classes. The more degenerate class is the set of product metrics which are also studied from an invariant point of view.

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## 1. Introduction

The study of warped product metrics is of interest in general relativity because they include a wide variety of physically relevant solutions of Einstein equations. For instance, spherically symmetric spacetimes, Lemaître-Friedmann-Robertson-Walker universes and static metrics are warped products.

Carot and da Costa proposed [1] an invariant characterization of warped product spacetimes and a classification scheme for them. They also studied the isometry group that each class can admit. Later, the energy tensor algebraic types compatible with these classes was analyzed [2]. Recent studies about warped product spacetimes concern Ricci collineations [3], parallel transport [4] and causal structure [5].

Besides the metric tensor, the characterization conditions given in [1] involve vector fields and functions that are not given in terms of the metric tensor. Consequently, a characterization of warped spacetimes built with *intrinsic* (depending solely on the metric tensor) and *explicit* conditions is not yet known. The goal of this paper is to obtain such an intrinsic and explicit characterization.

The intrinsic labeling of metrics starts with the beginning of the Riemannian geometry [6, 7, 8, 9, 10]. Cartan [11] showed that a metric may be characterized in terms of the Riemann tensor and its covariant derivatives. Brans [12] introduced the Cartan invariant scheme in general relativity, and Karlhede [13] developed a method to study the equivalence of two metric tensors. It is worth pointing out that the covariant determination of the underlying geometry of the Weyl and Ricci tensors is a necessary tool to characterize spacetimes intrinsically. This study can be found in [14] for the Ricci tensor and in [15] for the Weyl tensor.

The intrinsic and explicit characterization of spacetimes is interesting not only from the conceptual point of view but also from the practical one. Indeed, it provides an algorithmic way to test if a metric tensor, given in an arbitrary coordinate system, is a specific solution of Einstein equations. This is a powerful tool for subsequent applications. Thus, intrinsic labeling of a solution can be useful in obtaining a fully algorithmic characterization of the initial data which correspond to this solution. For instance, the algorithmic characterization of the Schwarzschild initial data [16] has been achieved by applying our intrinsic and explicit labeling of the Schwarzschild geometry [17]. Recently, we have accomplished an analogous study for the Kerr black hole [18] which will undoubtedly help to obtain algorithmic Kerr initial data. Similar approaches, essentially based on the accurate inquiry of the underlying geometry of the Weyl and Ricci tensors, have allowed us to intrinsically characterize several families of spacetimes which are interesting from the physical and/or geometrical point of view [15, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28].

The conditions that we can find in all the intrinsic characterizations quoted above are local restrictions on the curvature tensor. Here we also study local properties and consider *locally warped products*, that is,  $n$ -dimensional Riemann spaces satisfying that in a neighborhood of each point there is a coordinate system  $\{x^\alpha\}$  such that the metric line element can be written:

$$ds^2 = h_{ij}^1(x^k)dx^i dx^j + e^{2\lambda(x^k)} h_{AB}^2(x^C)dx^A dx^B, \quad (1)$$

where  $i, j, k = 1, \dots, p$ , and  $A, B, C = p+1, \dots, n$ . The two factors  $h^1$  and  $h^2$  are  $p$  and  $n-p$  dimensional metrics, respectively, and  $\lambda$  is the *warping factor*. For short, we write  $g = h^1 + e^{2\lambda}h^2$  and say that  $g$  is a *warped metric*. For a formal definition and basic properties of warped product metrics see [29].

In relativity we can consider 1+3 and 2+2 warped spacetimes. The 1+3 case can be characterized in terms of the kinematic properties of a unitary vector field [1]. For 2+2 warped spacetimes the known characterization is not as neat because involves two null vectors and also the warping factor  $\lambda$  [1, 2].

Here we restrict ourselves to study the 2+2 warped spacetimes and we improve our understanding of them by giving several equivalent geometric conditions that characterize them: in terms of the projector  $v$  or of the canonical 2-form  $U$  associated with the 2-planes of the warped product, or imposing the existence of a Killing-Yano tensor subject to complementary properties.

The principal planes, and consequently the invariant tensors  $v$  and  $U$ , are closely linked with the Ricci and/or Weyl tensors and could be obtained using the covariant approach given in [14, 15]. This fact enables us to give an intrinsic labeling of the the non conformally flat 2+2 warped spacetimes imposing conditions on explicit Weyl concomitants. The results that we present here will allow us to deal with an old problem stated by Takeno [30]: the intrinsic an explicit characterization of the spherically symmetric spacetimes. This question will be analyzed elsewhere [31] by considering these metrics as particular 2+2 warped spacetimes.

A 2+2 warped spacetime is conformal to a 2+2 product spacetime with particular conformal factor. This fact determines the approach used here to obtain our results. In Section 2 we analyze the product metrics: we give several equivalent conditions which characterize them, we summarize known properties of their curvature tensor and we offer an algorithmic way to detect them. In Section 3 we present a similar study for the conformally 2+2 product metrics and we give an invariant classification of these spacetimes. In Section 4 we show that the 2+2 warped spacetimes correspond to two of these invariant classes, the t-warped and the s-warped metrics. As a consequence, we obtain the complementary conditions for a conformally 2+2 product metric to be a warped spacetime. Moreover we see that, for type D metrics, these conditions impose restrictions on the Weyl principal 2-form arriving thus to an intrinsic and explicit characterization of these spacetimes. The algorithmic nature of our results allows us to present a flow chart in Section 5, which distinguishes every invariant class of conformally 2+2 product metrics and, in particular, the t-warped and s-warped spacetimes. Finally, in Section 6 we comment on our main results and work underway.

In this paper we work on an oriented spacetime with a metric tensor  $g$  of signature  $\{-, +, +, +\}$ . The Riemann, Ricci and Weyl tensors are defined as given in [32] and denoted, respectively, by  $\text{Riem}(g)$ ,  $R = \text{Ric}(g)$  and  $W = W(g)$ . For the metric product of two vectors we write  $(x, y) = g(x, y)$ , and we put  $x^2 = g(x, x)$ . Other basic notation used in this work is summarized in Appendix A. In Appendix B we introduce basic concepts on 2+2 almost-product structures and summarize their invariant classification. Finally, Appendix C presents the covariant determination of the canonical 2-form of a type D Weyl tensor.

## 2. 2+2 product spacetimes

In a 2+2 *product spacetime* the metric tensor takes (locally) the expression  $\hat{g} = \hat{v} + \hat{h}$ , where  $\hat{v}$  and  $\hat{h}$  denote two 2-dimensional metrics, Lorentzian and Riemannian, respectively. We have two *principal planes*, a time-like one  $V$ , with projector  $\hat{v}$ , and its orthogonal complement, the space-like plane  $H$ , with projector  $\hat{h}$ . The *canonical 2-form* of the 2+2 product spacetime is the volume element  $\hat{U}$  of the time-like plane,  $\hat{U}^2 = \hat{v}$ .

### 2.1. Invariant characterization

In a 2+2 product spacetime the generalized second fundamental forms  $Q_{\hat{v}}$  of  $V$  and  $Q_{\hat{h}}$  of  $H$  vanish, that is, the principal planes define an integrable and totally geodesic 2+2 almost-product structure. Then, one says that  $(\hat{v}, \hat{h})$  is a *product structure* and that  $\hat{g}$  is a 2+2 *product metric*. In Appendix B we can find the basic concepts and properties on almost-product structures. From them we can easily obtain the following characterizations of the 2+2 product metrics.

**Proposition 1** *A spacetime is (locally) a 2+2 product if, and only if, one of the following equivalent conditions holds:*

(i) *There exists a 2+2 product structure  $(\hat{v}, \hat{h})$ :  $Q_{\hat{v}} = 0$ ,  $Q_{\hat{h}} = 0$ .*

(ii) *There exists a covariantly constant simple and unitary 2-form  $\hat{U}$ :*

$$\text{tr } \hat{U}^2 = 2, \quad \hat{U} \cdot * \hat{U} = 0, \quad \nabla \hat{U} = 0. \quad (2)$$

(iii) *There exists a covariantly constant 2-projector  $\hat{v}$ :*

$$\hat{v}^2 = \hat{v}, \quad \text{tr } \hat{v} = 2, \quad \nabla \hat{v} = 0. \quad (3)$$

(iv) *There exists a covariantly constant traceless involutive 2-tensor  $\hat{\Pi}$ :*

$$\hat{\Pi}^2 = \hat{g}, \quad \text{tr } \hat{\Pi} = 0, \quad \nabla \hat{\Pi} = 0. \quad (4)$$

(v) *There exists a simple and unitary Killing-Yano tensor of order two.*

The 2-form  $\hat{U}$  mentioned in (ii) is the canonical 2-form of the product structure cited in (i), and its square is the projector  $\hat{v}$  cited in (iii). In this statement we can replace  $\hat{v}$  by the projector  $\hat{h}$  on the space-like plane. The 2-tensor  $\hat{\Pi}$  quoted in (iv) is the structure tensor  $\hat{\Pi} = \hat{v} - \hat{h}$ . The Killing-Yano tensor named in (v) is  $\hat{U}$ : the Killing-Yano equation implies a vanishing covariant derivative under the simple and unitary restrictions. Obviously,  $\hat{v} = \hat{U}^2$  is a Killing tensor.

All the characterizations in the proposition above are invariant properties of the spacetime. Nevertheless, these conditions are not intrinsic because they involve, besides the metric tensor, other geometric quantities, namely, the 2-tensors  $\hat{U}$ ,  $\hat{v}$  or  $\hat{\Pi}$ . In order to obtain a fully intrinsic characterization we must study the curvature tensor of a product metric.

### 2.2. Ricci and Weyl tensors

The Weyl and Ricci tensor of a product metric have been studied in detail by many authors. The important fact is that they exclusively depend on the curvatures of the two 2-dimensional metrics  $\hat{v}$  and  $\hat{h}$ . We summarize some properties that we need here.

**Proposition 2** *Let  $\hat{g} = \hat{v} + \hat{h}$  be a 2+2 product metric,  $X$  and  $Y$  the Gauss curvature of  $\hat{v}$  and  $\hat{h}$ , respectively, and  $\hat{U}$  such that  $\hat{U}^2 = \hat{v}$ . It holds:*

(i)  $\text{Riem}(\hat{g}) = -X \hat{U} \otimes \hat{U} + Y * \hat{U} \otimes * \hat{U}$ .

- (ii)  $\text{Ric}(\hat{g}) = \text{Ric}(\hat{v}) + \text{Ric}(\hat{h}) = X \hat{v} + Y \hat{h}$ .
- (iii)  $\text{tr Ric}(\hat{g}) = 2(X + Y)$ .
- (iv)  $W(\hat{g}) = \hat{\rho}(3\hat{S} + \hat{G})$ ,  $\hat{\rho} \equiv -\frac{1}{6}(X + Y)$ ,  $\hat{S} \equiv \hat{U} \otimes \hat{U} - * \hat{U} \otimes * \hat{U}$ ,  $\hat{G} \equiv \frac{1}{2} \hat{g} \otimes \hat{g}$ .

As a direct consequence of this proposition we have:

**Corollary 1** *In a 2+2 product spacetime:*

- (i) *The Weyl tensor is Petrov-Bel type D (or O, i.e. conformally flat) being  $\hat{\rho} = -\frac{1}{6}(X + Y)$  the double eigenvalue. In the type D case, the Weyl principal planes are the principal planes of the product.*
- (ii) *The Ricci tensor is Segre type  $[(11)(11)]$  (or  $[(1111)]$ , i.e. Einstein space), with eigenvalues  $X$  and  $Y$ . In the first case, the Ricci eigen-planes are the principal planes of the product.*
- (iii) *The scalar curvature vanishes if, and only if, the spacetime is conformally flat.*

### 2.3. Intrinsic and explicit characterization

Proposition 2 shows that the 2-form  $\hat{U}$  quoted in point (ii) of proposition 1 is the principal 2-form of a type D Weyl tensor. Moreover, the 2-tensors  $\hat{v}$  and  $\hat{\Pi}$  quoted in points (iii) and (iv) are, respectively, the projector on the time-like eigen-plane of a Ricci tensor of type  $[(11)(11)]$  and the associated structure tensor. Consequently, these invariant characterizations become intrinsic and we can state:

**Proposition 3** *A spacetime (which is not an Einstein space) is a 2+2 product if, and only if, its Ricci tensor is of type  $[(11)(11)]$  and the projector on the Ricci time-like eigen-plane is covariantly constant.*

**Proposition 4** *A non conformally flat spacetime is a 2+2 product if, and only if, it is Petrov-Bel type D and the principal 2-form of the Weyl tensor is covariantly constant.*

It is worth remarking that the two propositions above cover all the metrics with non constant sectional curvature. Moreover, as a consequence of corollary 1, a product metric with constant sectional curvature is, necessarily, flat. Note that a non flat product metric admits a unique factorization. If the metric is flat, we have a factorization for every flat plane.

The intrinsic conditions in propositions 3 or 4 may be tested for a given metric tensor if we are able to impose the algebraic type and to determine the Weyl principal 2-form or the Ricci eigen-planes. Moreover, if we know a covariant expressions of these geometric elements in terms of the Weyl or Ricci tensors, we can obtain an intrinsic and explicit labeling of the 2+2 product spacetimes.

Let  $\hat{R} \equiv \text{Ric}(\hat{g})$  and  $\hat{r} \equiv \text{tr Ric}(\hat{g})$  be the Ricci tensor and the scalar curvature of the product metric  $\hat{g}$ , and  $N$  the Ricci trace-less part,  $N = \hat{R} - \frac{1}{4} \hat{r} g$ . Then,  $\hat{R}$  is of type

[(11)(11)] if, and only if,  $\text{tr } N^2 > 0$  and  $N^2 = \frac{1}{4} \text{tr } N^2 g$ . Moreover, the structure tensor  $\Pi$  of the Ricci eigen-planes can be obtained as:

$$\Pi = \epsilon \frac{2}{\sqrt{\text{tr } N^2}} N, \quad \epsilon \equiv \frac{N(x, x)}{|N(x, x)|}, \quad (5)$$

where  $x$  is an arbitrary time-like vector. Then, under the above algebraic restrictions, the necessary and sufficient condition for the metric to be a 2+2 product is  $\nabla \Pi = 0$ . Consequently, substituting expression (5) of  $\Pi$ , we obtain:

**Theorem 1** *A spacetime (which is not an Einstein space) is a 2+2 product if, and only if, the trace-less part  $N$  of the Ricci tensor,  $N = \hat{R} - \frac{1}{4} \hat{r} g \neq 0$ , satisfies:*

$$\text{tr } N^2 > 0, \quad N^2 = \frac{1}{4} \text{tr } N^2 g, \quad \nabla N = \frac{1}{2} d \ln[\text{tr } N^2] \otimes N. \quad (6)$$

Moreover the two metric factors can be obtained as  $\hat{v} = \frac{1}{2}(g + \Pi)$  and  $\hat{h} = \frac{1}{2}(g - \Pi)$ , where  $\Pi$  is given in (5).

Note that the third (differential) condition in (6) states that  $N$  is a recurrent tensor, that is, a vector field  $e$  exists such that

$$\nabla N = e \otimes N. \quad (7)$$

Moreover, under the two first (algebraic) conditions given in (6), equation (7) implies  $e = \frac{1}{2} d \ln[\text{tr } N^2]$ . Consequently, we can state.

**Corollary 2** *A spacetime (which is not an Einstein space) is a 2+2 product if, and only if, the trace-less part  $N$  of the Ricci tensor,  $N = \hat{R} - \frac{1}{4} \hat{r} g \neq 0$ , is of type [(11)(11)] and recurrent.*

We have shown in [17] that the Weyl tensor  $W$  is type D with real eigenvalues if, and only if,  $\rho \equiv -(\frac{1}{12} \text{tr } W^3)^{\frac{1}{3}} \neq 0$  and  $\hat{S}^2 + \hat{S} = 0$ , where  $\hat{S} \equiv \frac{1}{3\rho}(W - \rho G)$ . Moreover  $\rho$  is the double Weyl eigenvalue and  $S$  depends on the principal 2-form  $\hat{U}$  as  $\hat{S} \equiv \hat{U} \otimes \hat{U} - * \hat{U} \otimes * \hat{U}$ . Then, can be easily shown that  $\hat{U}$  is covariantly constant if, and only if,  $\nabla \hat{S} = 0$ . Consequently, substituting the expression of  $\hat{S}$  in terms of the Weyl tensor and applying proposition 4, we obtain:

**Theorem 2** *A non conformally flat spacetime is a 2+2 product if, and only if, the Weyl tensor satisfies:*

$$\rho \equiv -\left(\frac{1}{12} \text{tr } W^3\right)^{\frac{1}{3}} \neq 0, \quad W^2 + \rho W - 2\rho^2 G = 0, \quad \nabla W = d \ln |\rho| \otimes W. \quad (8)$$

Moreover the two metric factors can be obtained as  $\hat{v} = \hat{U}^2$  and  $\hat{h} = \hat{g} - \hat{v}$ , where  $\hat{U} \equiv U[W]$  is given in Appendix C.

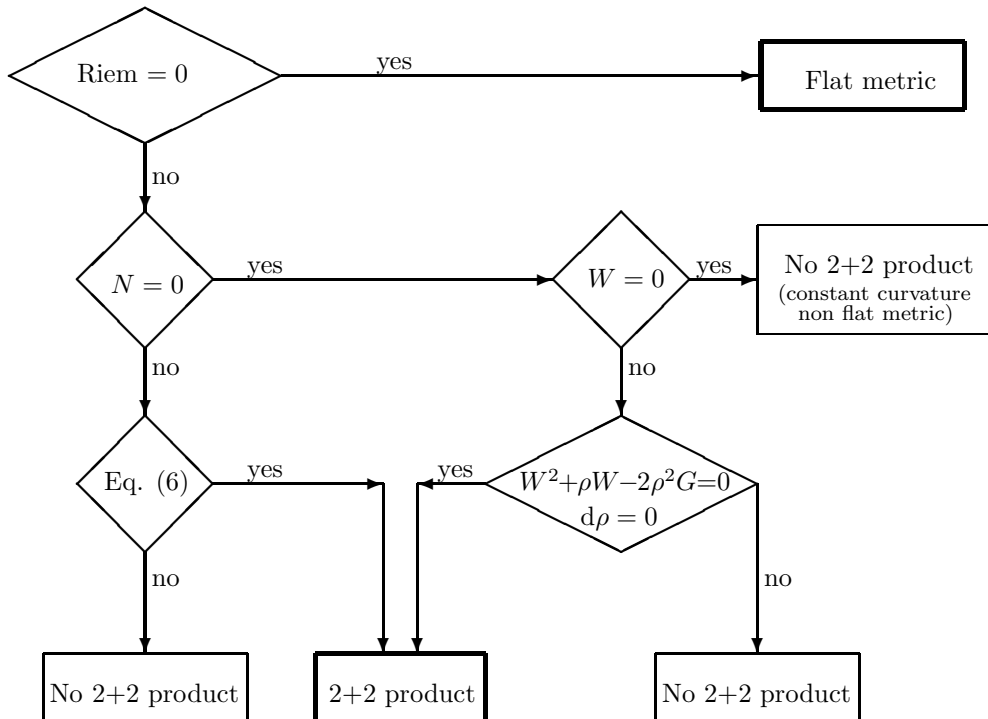
Note again that the third (differential) condition in (8) states that  $W$  is a recurrent tensor, that is, a vector field  $f$  exists such that

$$\nabla W = f \otimes W. \quad (9)$$

Moreover, under the two first (algebraic) conditions given in (8), equation (9) implies  $f = d \ln |\rho|$ . Consequently, we can state.

**Corollary 3** *A non conformally flat spacetime is a 2+2 product if, and only if, the Weyl tensor is Petrov-Bel type D with real eigenvalues and recurrent.*

The intrinsic and explicit characterizations given in theorems 1 and 2 allow us to build an algorithm to test whether a metric tensor is a 2+2 product. Note that if the spacetime is an Einstein space, the Cotton tensor vanishes,  $\nabla \cdot W = 0$ , and then  $\nabla U = 0$  if, and only if, the Weyl eigenvalue is constant. Then we arrive to the algorithm that we present below as a flow chart.



### 3. Conformally 2+2 product spacetimes

In a *conformally 2+2 product spacetime* the metric tensor takes the expression  $g = e^{2\lambda} \hat{g}$ , where  $\hat{g}$  is a (locally) 2+2 product metric,  $\hat{g} = \hat{v} + \hat{h}$ . The principal planes  $V$  and  $H$  have associated the projectors  $v = e^{2\lambda} \hat{v}$  and  $h = e^{2\lambda} \hat{h}$ . Then  $g = v + h$  and  $\Pi = v - h$  is the structure tensor. Taking into account the relations between two conformal metrics, we obtain the change of the generalized second fundamental forms:

**Lemma 1** *If  $g = v + h = e^{2\lambda} (\hat{v} + \hat{h})$ , we have*

$$Q_v = e^{2\lambda} [Q_{\hat{v}} - \hat{v} \otimes \hat{h} (d\lambda)], \quad Q_h = e^{2\lambda} [Q_{\hat{h}} - \hat{h} \otimes \hat{v} (d\lambda)].$$

Note that this lemma implies the conformal invariance of the umbilical and integrable properties.

#### 3.1. Invariant characterization

If we use the results of the previous section on product metrics and lemma above, and we take into account that  $\hat{v}^\beta_\alpha = v^\beta_\alpha$ , we have that the necessary and sufficient conditions

for a metric  $g$  to be conformal to a 2+2 product one is that  $g = v + h$  with

$$Q_v = -v \otimes h(d\lambda), \quad Q_h = -h \otimes v(d\lambda) \quad (10)$$

Moreover,  $Q_v$  and  $Q_h$  determine (and are determined by) the covariant derivatives of the canonical elements associated with the structure. Consequently, we obtain the following:

**Proposition 5** *A conformally 2+2 product spacetime is characterized by one of the following equivalent conditions:*

- (i) *There exists a 2+2 almost-product structure  $(v, h)$  and a function  $\lambda$  such that  $Q_v = -v \otimes h(d\lambda)$ ,  $Q_h = -h \otimes v(d\lambda)$ .*
- (ii) *There exists a simple and unitary 2-form  $U$  ( $U \cdot *U = 0$ ,  $\text{tr} U^2 = 2$ ) such that*

$$2 \nabla U = -U \oslash a + h \oslash U(b), \quad d(a + b) = 0, \quad (11)$$

*where  $v \equiv U^2$ ,  $h \equiv g - v$ ,  $a \equiv -*U(\nabla \cdot *U)$ ,  $b \equiv U(\nabla \cdot U)$ .*

- (iii) *There exists a traceless and involutive 2-tensor  $\Pi$  ( $\text{tr} \Pi = 0$ ,  $\Pi^2 = g$ ) such that*

$$2 \nabla \Pi = \Pi \overset{23}{\otimes} \Phi - g \overset{23}{\otimes} \Pi(\Phi), \quad d\Phi = 0, \quad \Phi \equiv \frac{1}{2} \Pi(\nabla \cdot \Pi). \quad (12)$$

- (iv) *There exists a simple conformal Killing-Yano tensor of order two.*

In proposition above,  $\oslash$  and  $\overset{23}{\otimes}$  denote, respectively, the anti-symmetrization and symmetrization in the second and third indexes of the tensorial product (see notation in Appendix A).

As happens in the 2+2 product case, all these geometrical elements are closely related to each other. The 2-tensor  $\Pi$  cited in (iii) is the structure tensor of the almost-product structure quoted in (i), and the 2-form  $U$  in (ii) is the associated canonical 2-form. In (ii) we can substitute the time-like 2-form  $U$  by the space-like one  $*U$ ,  $2 \nabla *U = -*U \oslash b + v \oslash *U(a)$ . Moreover  $\Phi = a + b = -2 d\lambda$ . The conformal Killing-Yano tensor cited in (iv) is  $e^\lambda U$ , and the proof of this point follows by considering the projections of the conformal Killing-Yano equation on the planes  $V$  and  $H$ . The traceless part of the square of this conformal Killing-Yano tensor is the conformal Killing tensor  $e^{2\lambda} \Pi$ .

Carot and Tupper [33] have classified the conformally 2+2 product spacetimes according to their conformal algebra. Moreover, they give an invariant characterization which imposes conditions on two null vectors and also involves the conformal factor. However, the invariant characterizations given in proposition 5 involve geometric elements linked with the curvature tensor. Then, we can obtain an intrinsic and explicit characterization of a conformally 2+2 product metric if we know its Ricci and Weyl tensors.



### 3.2. Ricci and Weyl tensors

The Weyl tensor  $W^\alpha_{\beta\mu\nu}$  is conformal invariant. Then only the Weyl eigenvalue and the Ricci tensor change. The following proposition summarizes these known results.

**Proposition 6** *Let  $g = e^{2\lambda}(\hat{v} + \hat{h})$  be a conformally 2+2 product metric and let us denote  $\hat{\nabla}$  the covariant derivative associated to the metric  $\hat{g} = \hat{v} + \hat{h}$ , and  $X, Y$  the Gauss curvature of  $\hat{v}$  and  $\hat{h}$ . Then,*

- (i)  $\text{Ric}(g) = X\hat{v} + Y\hat{h} - 2[\hat{\nabla}d\lambda - d\lambda \otimes d\lambda] - [\hat{\Delta}\lambda + 2\hat{g}(d\lambda, d\lambda)]\hat{g}$
- (ii) *The Weyl tensor is type D (or O) with double eigenvalue  $\rho \equiv -\frac{1}{6}e^{-2\lambda}(X + Y)$ . In the type D case, the canonical 2-form  $U$ ,  $U^2 = v$ , is the principal 2-form of the Weyl tensor which takes the expression:*

$$W = \rho(3S + G), \quad S \equiv U \otimes U - *U \otimes *U, \quad G \equiv \frac{1}{2}g \oslash g. \quad (13)$$

### 3.3. Intrinsic and explicit characterization

Proposition 6 shows that the algebraic type of the Ricci tensor of a conformally 2+2 metric strongly depends on the Hessian of the conformal factor  $\lambda$ . Thus, we have no restrictions on the Segre type and the Ricci tensor is, generically, algebraically arbitrary.

On the other hand, the Weyl tensor is, necessarily, type D or O. In the conformally flat case we can only detect the principal structure if one considers particular conformal factors that restrict the Ricci tensor type. But we do not consider here these particular situations. Nevertheless, now we acquire an intrinsic characterization of the non conformally flat metrics conformal to a 2+2 product because of the 2-form  $U$  quoted in point (ii) of proposition 5 is the Weyl principal 2-form.

**Proposition 7** *A non conformally flat spacetime is conformal to a 2+2 product if, and only if, it is Petrov-Bel type D and the Weyl principal 2-form satisfies (11).*

The differential conditions (11) can be tested if we know the Weyl principal 2-form  $U$ . In making more explicit these conditions we can consider in a first step the Weyl concomitant  $S \equiv \frac{1}{3\rho}(W - \rho G)$  which depends on  $U$  as  $S = U \otimes U - *U \otimes *U$ . Then, a straightforward calculation allows us to write (11) in terms of  $S$ . Moreover, taking into account the characterization of the type D Weyl tensors with real eigenvalues [17], one arrives to:

**Theorem 3** *A non conformally flat spacetime is conformal to a 2+2 product if, and only if, the Weyl tensor satisfies:*

$$\rho \equiv -\left(\frac{1}{12}\text{tr} W^3\right)^{\frac{1}{3}} \neq 0, \quad S^2 + S = 0, \quad S \equiv \frac{1}{3\rho}(W - \rho G); \quad (14)$$

$$2\nabla \cdot S + 3S(\nabla \cdot S) - g \oslash \Phi = 0, \quad d\Phi = 0, \quad \Phi \equiv \text{tr}[S(\nabla \cdot S)]. \quad (15)$$

Moreover the two metric factors can be obtained as  $v = U^2$  and  $h = g - v$ , where  $U \equiv U[W]$  is given in Appendix C, and the conformal factor  $\lambda$  is a function satisfying  $-2d\lambda = \Phi$ .

The notation used in this theorem is explained in detail in Appendix A. Note that  $\nabla \cdot S$

and  $\Phi$  are, respectively, a vector valued 2-form and a 1-form.

Finally, if one introduces the 1-form  $\omega = 3\rho(3\rho\Phi - 2d\rho)$  and substitutes the expression of  $S$  in terms of the Weyl tensor in the theorem above, one obtains:

**Theorem 4** *A non conformally flat spacetime is conformal to a 2+2 product if, and only if, the Weyl tensor satisfies:*

$$\rho \equiv -\left(\frac{1}{12} \text{tr } W^3\right)^{\frac{1}{3}} \neq 0, \quad W^2 + \rho W - 2\rho^2 G = 0; \quad (16)$$

$$\rho \nabla \cdot W + W(\nabla \cdot W) - \frac{1}{3} g \otimes \omega = 0, \quad d\left(\frac{1}{\rho^2} \omega\right) = 0, \quad \omega \equiv \text{tr } [W(\nabla \cdot W)]. \quad (17)$$

Note that when the Cotton tensor vanishes,  $\nabla \cdot W = 0$ , the differential conditions (17) identically hold. Consequently, we recover a result given in [23]: every type D metric with vanishing Cotton tensor is conformal to a product metric.

### 3.4. Invariant classification of conformally 2+2 product metrics

When a family of spacetimes has an outlined 2+2 almost-product structure we can classify them by considering the invariant classification of this structure (see Appendix B). For example, elsewhere [23] we have classified type D metrics attending to the differential properties of the Weyl principal structure, and we have shown that only 16 of the 64 possible classes are compatible with the vacuum condition.

The principal structure of a conformally 2+2 product spacetime is umbilical and integrable. Consequently, following the notation given in Appendix A, necessarily the structure is of type  $\binom{00}{00}$ , that is, only the classes  $\binom{001}{001}$ ,  $\binom{001}{000}$ ,  $\binom{000}{001}$  and  $\binom{000}{000}$  are possible. For short we will write  $\binom{r}{s}$  to indicate the conformally 2+2 product metrics of class  $\binom{00r}{00s}$ :

**Definition 1** *We say that a conformally 2+2 product metric is of:*

*class  $\binom{0}{0}$  if both principal planes are minimal (product metrics).*

*class  $\binom{0}{1}$  if the time-like principal plane is minimal and the space-like one is not.*

*class  $\binom{1}{0}$  if the space-like principal plane is minimal and the time-like one is not.*

*class  $\binom{1}{1}$  if none of the principal planes is minimal.*

The classification given in the definition above is based on the minimal character of the principal planes and imposes differential conditions on the canonical 2-form  $U$ . Nevertheless, every class can also be characterized in terms of the gradient of the conformal factor. Indeed, as a consequence of proposition 5, and taking into account that  $-2d\lambda = a + b$ , we obtain:

**Proposition 8** *For a conformally 2+2 product metric  $g = e^{2\lambda}(\hat{v} + \hat{h})$ , let  $U$  be the canonical 2-form and  $a \equiv -*U(\nabla \cdot *U)$ ,  $b \equiv U(\nabla \cdot U)$ . Then, the metric is:*

*class  $\binom{0}{0}$  iff  $a = b = 0$  iff  $\lambda$  is constant ( $d\lambda = 0$ ).*

*class  $\binom{0}{1}$  iff  $a = 0$ ,  $b \neq 0$  iff  $d\lambda$  lies on the time-like principal plane ( $d\lambda \neq 0$ ,  $*U(d\lambda) = 0$ ).*

class  $\binom{1}{0}$  iff  $a \neq 0$ ,  $b = 0$  iff  $d\lambda$  lies on the space-like principal plane ( $d\lambda \neq 0$ ,  $U(d\lambda) = 0$ ).

class  $\binom{1}{1}$  iff  $a \neq 0$ ,  $b \neq 0$  iff  $d\lambda$  does not lie on a principal plane ( $U(d\lambda) \neq 0$ ,  $*U(d\lambda) \neq 0$ ).

#### 4. 2+2 warped spacetimes

We can consider two different types of 2+2 warped spacetimes depending on the Lorentzian or Riemannian character of the warped factor  $h_2$  (see expression (1)). In the first case the warped plane is time-like, the metric tensor takes the expression  $g = e^{2\lambda}\hat{v} + h$ , with  $\hat{v}(d\lambda) = 0$ , and we say that the spacetime is *t-warped*. In the second case the warped plane is space-like, the metric tensor takes the expression  $g = v + e^{2\lambda}\hat{h}$ , with  $\hat{h}(d\lambda) = 0$ , and we say that the spacetime is *s-warped*. Here,  $\hat{v}$  and  $v$  denote two 2-dimensional Lorentzian metrics, and  $h$  and  $\hat{h}$  denotes two 2-dimensional Riemannian metrics. In [2] two subclasses of s-warped metrics were considered depending on the null or non null character of  $d\lambda$ . But in our study this distinction plays no role.

If for a 2+2 t-warped (respectively, s-warped) metric we define  $\hat{h} = e^{-2\lambda}h$  (respectively,  $\hat{v} = e^{-2\lambda}v$ ), we obtain that a 2+2 warped metric is conformal to a 2+2 product metric,  $g = e^{2\lambda}(\hat{v} + \hat{h})$ , with  $\hat{v}(d\lambda) = 0$  (respectively,  $\hat{h}(d\lambda) = 0$ ). Then, from proposition 8, we easily obtain:

**Proposition 9** *The 2+2 t-warped (respectively, s-warped) spacetimes are the conformally 2+2 product spacetimes of class  $\binom{1}{0}$  (respectively,  $\binom{0}{1}$ ).*

##### 4.1. Invariant characterization

As a consequence of proposition 9 we can characterize the warped spacetimes by adding the conditions that label classes  $\binom{1}{0}$  or  $\binom{0}{1}$  (see proposition 8) to the conditions given in section 4 that characterize the metrics conformal to 2+2 product. Indeed, a simple reasoning leads to:

**Proposition 10** *A 2+2 t-warped spacetime is characterized by one of the following equivalent conditions:*

- (i) *There exists a 2+2 almost product structure  $(v, h)$  and a function  $\lambda$  such that  $Q_v = -v \otimes d\lambda$ ,  $Q_h = 0$ .*
- (ii) *There exists a simple, time-like and unitary 2-form  $U$  ( $U \cdot *U = 0$ ,  $\text{tr } U^2 = 2$ ) such that*

$$2 \nabla U = -U \otimes a, \quad da = 0, \quad a \equiv - *U(\nabla \cdot *U) \neq 0. \quad (18)$$

- (iii) *There exists a time-like 2-projector  $v$  ( $v^2 = v$ ,  $\text{tr } v = 2$ ,  $v(x, x) < 0$ , where  $x$  is an arbitrary time-like vector) such that*

$$2 \nabla v = v \otimes a, \quad da = 0, \quad a \equiv \nabla \cdot v \neq 0. \quad (19)$$

(iv) There exists a simple and time-like Killing-Yano tensor of order two.

**Proposition 11** *A 2+2 s-warped spacetime is characterized by one of the following equivalent conditions:*

(i) There exists a 2+2 almost product structure  $(v, h)$  and a function  $\lambda$  such that  $Q_v = 0$ ,  $Q_h = -h \otimes d\lambda$ .

(ii) There exists a simple, time-like and unitary 2-form  $U$  ( $U \cdot *U = 0$ ,  $\text{tr } U^2 = 2$ ) such that

$$2 \nabla *U = - *U \oslash b, \quad db = 0, \quad b \equiv U(\nabla \cdot U) \neq 0. \quad (20)$$

(iii) There exists a space-like 2-projector  $h$  ( $h^2 = h$ ,  $\text{tr } h = 2$ ,  $h(x, x) \geq 0$ , where  $x$  is an arbitrary time-like vector) such that

$$2 \nabla h = h \overset{23}{\otimes} b, \quad db = 0, \quad b \equiv \nabla \cdot h \neq 0. \quad (21)$$

(iv) There exists a simple and space-like Killing-Yano tensor of order two.

All the geometrical elements in the proposition 10 (respectively, 11) above are closely related to each other. The 2-form  $U$  of (ii) is the canonical 2-form of the structure cited in (i), and the 2-tensor in (iii) is  $v = U^2$  (respectively,  $h = g - v$ ). Moreover,  $a = -2d\lambda$  (respectively,  $b = -2d\lambda$ ). The Killing-Yano tensor cited in (iv) is  $e^\lambda U$  (respectively,  $e^\lambda *U$ ). The square of this Killing-Yano tensor is the Killing tensor  $e^{2\lambda}v$  (respectively,  $e^{2\lambda}h$ ).

#### 4.2. Ricci and Weyl tensors

Now, to obtain an intrinsic and explicit characterization from the above invariant properties, we analyze the Ricci and Weyl tensors of a 2+2 warped metric.

A 2+2 warped spacetime is a conformally 2+2 product with particular (non constant) conformal factor  $\lambda$ . Then, the Ricci tensor is given in proposition 6. Now, for a t-warped (respectively, s-warped) spacetime we have  $U(d\lambda) = 0$  (respectively,  $*U(d\lambda) = 0$ ) and, consequently,  $\hat{\nabla} d\lambda - d\lambda \otimes d\lambda$  is a 2-tensor of the space-like plane  $H$  (respectively, time-like plane  $V$ ). Then, we have the following known result (see, for example, [2]):

**Proposition 12** *In a 2+2 t-warped (respectively, s-warped) spacetime the time-like principal plane (respectively, space-like plane) is an eigen-plane of the Ricci tensor.*

When the Ricci eigen-plane quoted in the proposition above corresponds to a double eigenvalue (Segre types [(11)11] or [11(11)]), the projectors  $v$  or  $h$  can be covariantly obtained from the Ricci tensor and, then, the invariant characterizations given in point (iii) of propositions 10 or 11 become intrinsic and explicit. Nevertheless, there are Segre types admitting a time-like or a space-like eigen-plane that are more degenerate. In these cases the intrinsic labeling of the 2+2 warped spacetimes using the Ricci tensor requires a more detailed analysis that will be considered elsewhere [34].

On the other hand, as also stated in proposition 6, the metric is Petrov-Bel type D (or O). In the type D case the Weyl tensor takes the expression (13) where  $\rho \equiv -\frac{1}{6}e^{-2\lambda}(X + Y)$  is the double eigenvalue and  $U$  is the Weyl principal 2-form which satisfies  $U^2 = v$ . Consequently, for non conformally flat 2+2 warped spacetimes we can obtain an intrinsic labeling.

#### 4.3. Intrinsic and explicit characterization

Propositions 8 and 9 imply that the 2+2 t-warped (respectively, s-warped) spacetimes are the conformally 2+2 product ones subject to the complementary restriction  $b = v(\Phi) = 0$  (respectively,  $a = h(\Phi) = 0$ ), a condition which is equivalent to  $U(\Phi) = 0$  (respectively,  $*U(\Phi) = 0$ ).

On the other hand, for the type D metrics ( $W \neq 0$ ) the double 2-form  $S$  given in (13) satisfies  $*S \equiv U \tilde{\otimes} *U$ , and consequently,  $*S(\Phi; \Phi) = U(\Phi) \tilde{\otimes} *U(\Phi)$ . Thus, we arrive to the following:

**Proposition 13** *A (non conformally flat) conformally 2+2 product is a 2+2 warped spacetime if, and only if, the Weyl tensor satisfies  $*S(\Phi; \Phi) = 0$  where  $S \equiv \frac{1}{3\rho}(W - \rho G)$  and  $\Phi \equiv \text{tr}[S(\nabla \cdot S)] \neq 0$ .*

Note that we have  $*W = \rho(3*S - \eta)$ , and, consequently, we can substitute condition  $*S(\Phi; \Phi) = 0$  in proposition above by  $*W(\Phi; \Phi) = 0$ .

To distinguish between t-warped and s-warped spacetimes we must add another condition. Under the assumption  $*S(\Phi; \Phi) = 0$  we have that  $U(\Phi) = 0$  or  $*U(\Phi) = 0$ . Then, the indefinite quadratic form  $Q = S(\Phi; \Phi) = U(\Phi) \otimes U(\Phi) - *U(\Phi) \otimes *U(\Phi)$  becomes semi-definite: negative for t-warped case and positive for s-warped spaces. Moreover, the sign of a semi-definite quadratic form is the sign of its trace with whatever elliptic metric associated with  $g$ ,  $2x \otimes x + g$ ,  $x$  being a unitary time-like vector. Thus, we have the following result.

**Proposition 14** *A non conformally flat 2+2 warped metric is:*

- (i) *t-warped if, and only if,  $Q = S(\Phi; \Phi)$  is semi-definite negative,  $2Q(x, x) + \text{tr } Q < 0$ .*
  - (ii) *s-warped if, and only if,  $Q = S(\Phi; \Phi)$  is semi-definite positive,  $2Q(x, x) + \text{tr } Q > 0$ .*
- where  $S$  and  $\Phi$  are given in proposition 13 and  $x$  is an arbitrary unitary time-like vector.

The two propositions above and theorem 3 that characterizes the conformally 2+2 product metrics lead to:

**Theorem 5** *For a non conformally flat metric  $g$  let  $\rho$ ,  $S$  and  $\Phi$  be the Weyl concomitants*

$$\rho \equiv -\left(\frac{1}{12} \text{tr } W^3\right)^{\frac{1}{3}} \neq 0, \quad S \equiv \frac{1}{3\rho}(W - \rho G), \quad \Phi \equiv \text{tr}[S(\nabla \cdot S)]. \quad (22)$$

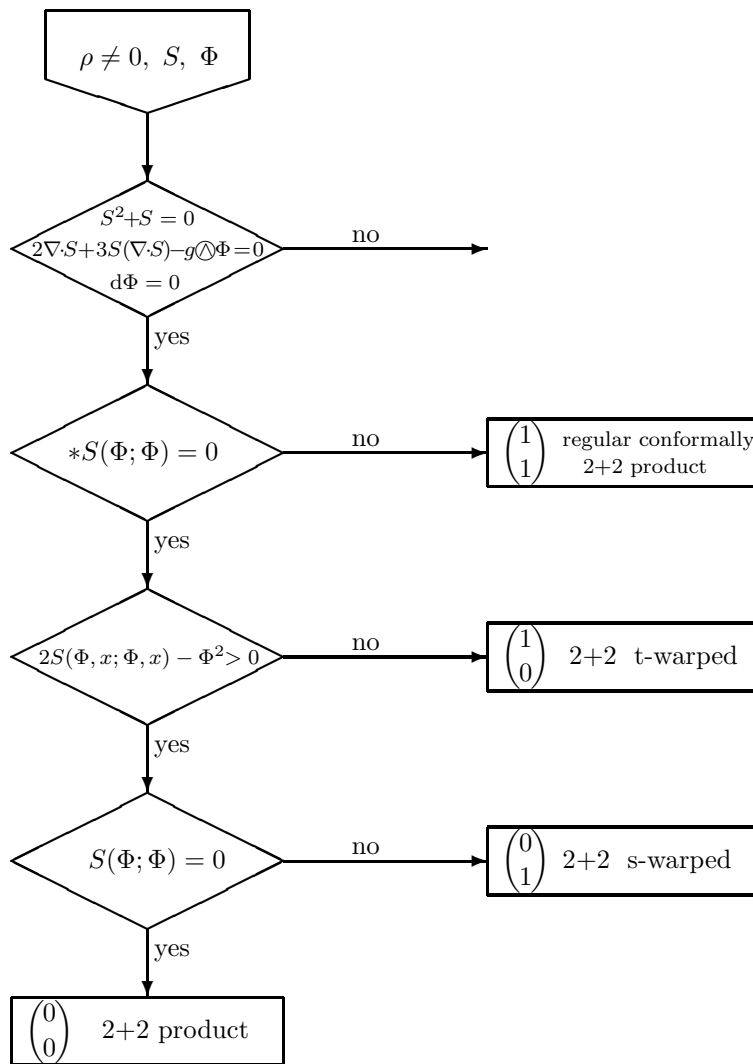
*Then,  $g$  is a 2+2 warped metric if, and only if, it satisfies:*

$$S^2 + S = 0, \quad 2\nabla \cdot S + 3S(\nabla \cdot S) - g \oslash \Phi = 0, \quad d\Phi = 0, \quad *S(\Phi; \Phi) = 0. \quad (23)$$

In addition, the metric is *t-warped* (respectively, *s-warped*), if, and only if,  $2S(\Phi, x; \Phi, x) - \Phi^2 < 0$  (respectively,  $2S(\Phi, x; \Phi, x) - \Phi^2 > 0$ ), where  $x$  is an arbitrary unitary time-like vector.

## 5. A summary in algorithmic form

Theorems 3 and 5 enable us to perform an algorithm which distinguishes, between all the non conformally flat metrics, the four classes of conformally 2+2 product spacetimes:  $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$  (product),  $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$  (t-warped),  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  (s-warped) and  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  (regular conformally product). The flow chart (see below) shows the role played by every condition in these theorems. The input data are the Weyl concomitants  $\rho = \rho(g)$ ,  $S = S(g)$  and  $\Phi = \Phi(g)$  defined in (22).



## 6. Discussion and work in progress

In this work we have acquired an intrinsic and explicit labeling of the non conformally flat 2+2 warped spacetimes (theorem 5). This result is based on the invariant characterization of these spacetimes in terms of a time-like and unitary 2-form  $U$  which

result to be the canonical 2-form of the (type D) Weyl tensor. In contrast, the invariant characterization given previously [1, 2] imposes conditions on two null vectors and it also involves the warping factor  $\lambda$ .

In [2] the warped spacetimes were classified taking into account the projection of the gradient of the warped factor on two null directions. Here we have shown that t-warped and s-warped spacetimes correspond to two classes of conformally product metrics when one applies a general classification of the 2+2 almost-product structures.

In the non conformally flat case we have labeled with intrinsic and explicit conditions (theorems 2, 3 and 5) the four compatible classes of this classification (product, t-warped, s-warped and regular conformally product). Thus, we were able to build an algorithm to distinguish them that we have presented as a flow chart.

Conformally flat product metrics have been fully studied here (theorem 1). Nevertheless, the intrinsic characterization of the 2+2 warped conformally flat spacetimes requires a more detailed analysis of the different Ricci algebraic types that will be considered elsewhere [34].

We have also shown in this work that the four classes quoted above can also be characterized in terms of a Killing-Yano or a conformal Killing-Yano tensor (see propositions 1, 5, 10 and 11). The existence of quadratic first integrals of the geodesic equation in these spacetimes has been shown in [2]. But here we have shown that, the existence of an algebraically special Killing-Yano or conformal Killing-Yano-tensor is a necessary and sufficient condition for a metric to be of a fixed class.

It is worth remarking that if a metric tensor, given in arbitrary coordinates, satisfies our characterization theorems, from the Riemann tensor we can obtain the geometric elements that appear in its canonical form, namely, the two factors  $v$  and  $h$  and the conformal (or warping) factor  $\lambda$  (see theorem 3).

Finally, we wish to comment that our results on s-warped metrics pave the way to acquiring the intrinsic labeling of the spherically symmetric spacetimes, a question that is considered elsewhere [31].

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## Appendix A. Notation

(i) **Products and other formulas involving 2-tensors  $A$  and  $B$ :**

(a) Composition as endomorphisms:  $A \cdot B$ ,

$$(A \cdot B)^{\alpha}_{\beta} = A^{\alpha}_{\mu} B^{\mu}_{\beta} \quad (\text{A.1})$$

(b) Square and trace as an endomorphism:

$$A^2 = A \cdot A, \quad \text{tr } A = A^{\alpha}_{\alpha}. \quad (\text{A.2})$$

- (c) Action on a vector  $x$ , as an endomorphism  $A(x)$ , and as a quadratic form  $A(x, x)$ :

$$A(x)^\alpha = A^\alpha_\beta x^\beta, \quad A(x, x) = A_{\alpha\beta} x^\alpha x^\beta. \quad (\text{A.3})$$

- (d) Exterior product as double 1-forms:  $A \otimes B$  is the double 2-form,

$$(A \otimes B)_{\alpha\beta\mu\nu} = A_{\alpha\mu} B_{\beta\nu} + A_{\beta\nu} B_{\alpha\mu} - A_{\alpha\nu} B_{\beta\mu} - A_{\beta\mu} B_{\alpha\nu}. \quad (\text{A.4})$$

- (e) Exterior product with a vector  $x$  as double 1-forms:  $A \otimes x$  is the vector-valued 2-form,

$$(A \otimes x)_{\alpha, \mu\nu} = A_{\alpha\mu} x_\nu - A_{\alpha\nu} x_\mu. \quad (\text{A.5})$$

- (f) Symmetrized tensorial product with a vector  $x$  as double 1-form:  $A \overset{23}{\otimes} x$  is the vector-valued symmetric 2-tensor,

$$(A \overset{23}{\otimes} x)_{\alpha, \mu\nu} = A_{\alpha\mu} x_\nu + A_{\alpha\nu} x_\mu. \quad (\text{A.6})$$

(ii) **Products and other formulas involving double 2-forms  $P$  and  $Q$ :**

- (a) Composition as endomorphisms of the 2-forms space:  $P \circ Q$ ,

$$(P \circ Q)^{\alpha\beta}_{\rho\sigma} = \frac{1}{2} P^{\alpha\beta}_{\mu\nu} Q^{\mu\nu}_{\rho\sigma} \quad (\text{A.7})$$

- (b) Square and trace as an endomorphism:

$$P^2 = P \circ P, \quad \text{Tr } P = \frac{1}{2} P^{\alpha\beta}_{\alpha\beta}. \quad (\text{A.8})$$

- (c) Action on a 2-form  $X$ , as an endomorphism  $P(X)$ , and as a quadratic form  $P(X, X)$ ,

$$P(X)_{\alpha\beta} = \frac{1}{2} P_{\alpha\beta}^{\mu\nu} X_{\mu\nu}, \quad P(X, X) = \frac{1}{4} P^{\alpha\beta\mu\nu} X_{\alpha\beta} X_{\mu\nu}. \quad (\text{A.9})$$

- (d) The Hodge dual operator is defined as the action of the, metric volume element  $\eta$  on a 2-form  $F$  and a double 2-form  $W$ :

$$*F = \eta(F), \quad *W = \eta \circ W. \quad (\text{A.10})$$

- (e) Action on two vectors  $x$  and  $y$ ,  $P(x; y)$ ,

$$P(x; y)_{\alpha\beta} = P_{\alpha\mu\beta\nu} x^\mu y^\nu. \quad (\text{A.11})$$

- (f) Action on a vector-valued 2-form  $Y$  as an endomorphism  $P(Y)$ ,

$$P(Y)_{\lambda, \alpha\beta} = \frac{1}{2} P_{\alpha\beta}^{\mu\nu} Y_{\lambda, \mu\nu}. \quad (\text{A.12})$$

- (g) The trace of a vector-valued 2-form  $Y$  is the 1-form  $\text{tr } Y$ ,

$$(\text{tr } Y)_\alpha = g^{\lambda\mu} Y_{\lambda, \mu\alpha}. \quad (\text{A.13})$$

## Appendix B. 2+2 almost-product structures

The generalized second fundamental form  $Q_v$  of a non-null p-plane  $V$  (set of vector fields generated by p independent vector fields) is the (2,1)-tensor:

$$Q_v(x, y) = h(\nabla_{v(x)} v(y)), \quad \forall x, y \quad (\text{B.1})$$



where  $v$  is the projector on  $V$  and  $h = g - v$ . Let us consider the invariant decomposition of  $Q_v$  into its antisymmetric part  $A_v$  and its symmetric part  $S_v \equiv S_v^T + \frac{1}{p}v \otimes \text{tr } S_v$ , where  $S_v^T$  is a traceless tensor:

$$Q_v = A_v + \frac{1}{p}v \otimes \text{tr } S_v + S_v^T \quad (\text{B.2})$$

The p-plane  $V$  is a *foliation* if, and only if,  $A_v = 0$ , and, similarly,  $V$  is said to be *minimal*, *umbilical* or *geodesic* if  $\text{tr } S_v = 0$ ,  $S_v^T = 0$  or  $S_v = 0$ , respectively.

On the spacetime, a 2+2 almost-product structure is defined by a time-like plane  $V$  and its space-like orthogonal complement  $H$ . Let  $v$  and  $h = g - v$  be the respective projectors and let  $\Pi = v - h$  be the *structure tensor*. A 2+2 spacetime structure is also determined by the *canonical* unitary 2-form  $U$ , volume element of the time-like plane  $V$ . Then, the respective projectors are  $v = U^2$  and  $h = -(U^2)^{\perp}$ .

The 2+2 almost-product structures can be classified by taking into account the invariant decomposition of the covariant derivative of the structure tensor  $\Pi$  or, equivalently, according to the foliation, minimal or umbilical character of each plane [23, 25]. Every principal 2-plane can be subject or not to three properties, so  $2^6 = 64$  classes can be considered.

**Definition 2** *Taking into account the foliation, minimal or umbilical character of each 2-plane we distinguish 64 different classes of 2+2 almost-product structures.*

We denote the classes as  $(\begin{smallmatrix} p & q & r \\ l & m & n \end{smallmatrix})$ , where the superscripts  $p, q, r$  take the value 0 if the time-like principal plane is, respectively, a foliation, a minimal or an umbilical plane, and they take the value 1 otherwise. In the same way, the subscripts  $l, m, n$  collect the foliation, minimal or umbilical nature of the space-like plane.

The most degenerated class that we can consider is  $(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix})$  which corresponds to a product structure, and the most regular one is  $(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})$  which means that neither  $V$  nor  $H$  are foliation, minimal or umbilical planes. We will put a dot in place of a fixed script (1 or 0) to indicate the set of metrics that cover both possibilities. So, for example, the metrics of type  $(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & \cdot \end{smallmatrix})$  are the union of the classes  $(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{smallmatrix})$  and  $(\begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix})$ ; or a metric is of type  $(\begin{smallmatrix} 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{smallmatrix})$  if the time-like 2-plane is a foliation.

We will say that a structure is integrable when both,  $V$  and  $H$  are a foliation, that is, of type  $(\begin{smallmatrix} 0 & \cdot & \cdot \\ 0 & \cdot & \cdot \end{smallmatrix})$ . We will say that the structure is minimal (respectively, umbilical) if both,  $V$  and  $H$  are minimal (respectively, umbilical), that is, of type  $(\begin{smallmatrix} \cdot & 0 & \cdot \\ \cdot & 0 & \cdot \end{smallmatrix})$  (respectively,  $(\begin{smallmatrix} \cdot & \cdot & 0 \\ \cdot & \cdot & 0 \end{smallmatrix})$ ).

## Appendix C. Covariant determination of the canonical 2-form $U$ of a type D Weyl tensor with real eigenvalues

The covariant determination of the canonical elements of the Weyl tensor for all the Petrov-Bel types was presented in [15] using the self-dual formalism. In a more recent paper [18] we have also given the expression of the canonical 2-form  $U$  of a type D Weyl tensor using real formalism. Here we particularize it for the case of real eigenvalues:

**Lemma 2** *For a Petrov-Bel type D Weyl tensor with real eigenvalue  $\rho \equiv -(\frac{1}{12} \text{tr } W^3)^{\frac{1}{3}}$ , the canonical 2-form  $U$  can be obtained as:*

$$U = U[W] \equiv \frac{1}{\chi \sqrt{\chi + \tilde{f}}} \left( (\chi + f) F + \tilde{f} * F \right); \quad F \equiv P(Z), \quad (\text{C.1})$$

where  $Z$  is an arbitrary 2-form and

$$P \equiv W - \rho G, \quad \chi \equiv \sqrt{f^2 + \tilde{f}^2}, \quad f \equiv \text{tr } F^2, \quad \tilde{f} \equiv \text{tr}(F \cdot * F). \quad (\text{C.2})$$

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